Feb. 24, 2014

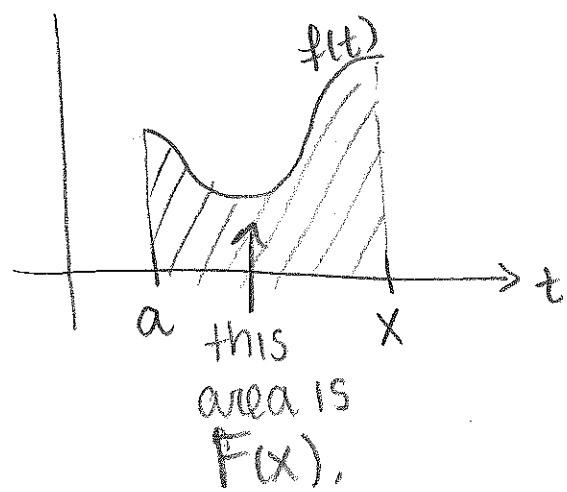
## The Fundamental Theorem of Calculus

(kind of a big deal)

Idea: derivatives and integrals are connected!

Set-up: $f(t)$ is a function, $a$ is a real #
Define a new function

$$F(x) = \int_a^x f(t)\,dt$$



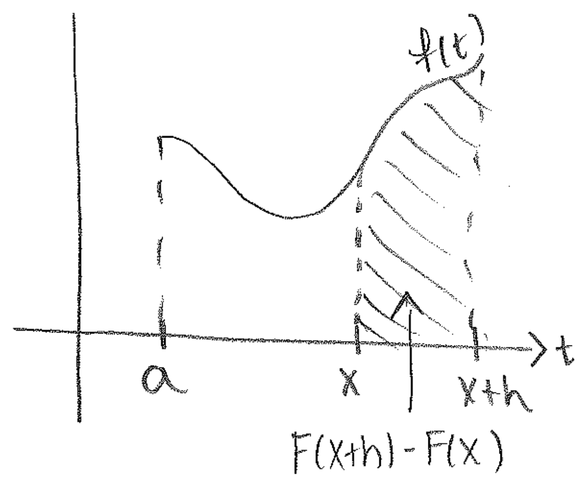

Let's consider the derivative of this func.

(Goal here - we are hoping the derivative undoes the integral sign)

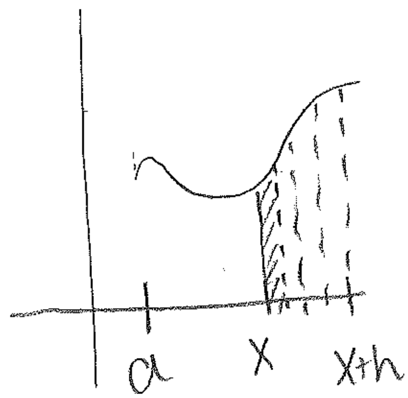
$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Numerator is a difference of areas



think about what's happen as $h \to 0$




area will get closer to the shape of the rectangle.

$\lim_{h\to 0} F(x+h)-F(x) \sim$ rectangle w/ width h and height $f(x)$

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x) = F'(x)$$

Fundamental Thm of Calculus

<u>Part I</u>: Suppose $f$ is a continuous function on an interval $I$ containing "a".

define $F(x) = \int_a^x f(t)\,dt$.

Then $F'(x) = f(x)$

*Moral: derivatives are the opposite of integrals!

<u>Example</u>: $\frac{d}{dx}\int_0^x (\ln t + t^2 \sin t)\,dt$

$= \ln x + x^2 \sin x$

Part II: Let $G(x)$ be an antiderivative of $f$ on $I$

Then for any $b$ in $I$

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

* Moral: to calculate precise area, we just need to take antiderivatives!

Example: $\int_0^3 \underbrace{x^2}_{\text{first find an antiderivative}} dx = \underbrace{\frac{x^3}{3}}_{G(x)}\Big|_0^3 = \underbrace{\frac{(3)^3}{3} - \frac{(0)^3}{3}}_{G(b) - G(a)} = 9$

Reality Check of Part II:

Notice $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ by part I.

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\left(F(x) - \underbrace{F(a)}_{\text{constant}}\right)$$

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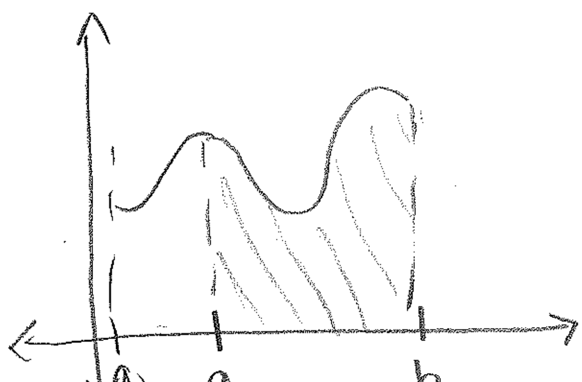
$$f(x) \qquad\qquad\quad f(x)$$



Examples:

(1) $\int_0^{\pi/4} \sin x\,dx = -\cos x \Big|_0^{\pi/4} = (-\cos \pi/4) - (-\cos 0)$

$= \boxed{-\sqrt{2}/2 + 1}$

use part II

(2) $\frac{d}{dx}\int_1^x t^2\,dt = x^2$ by part I

(3) $\underbrace{\frac{d}{dx}\int_1^{x^2} t^3\,dt}_{} = \underbrace{(x^2)^3}_{\substack{\text{derivative}\\\text{of outside}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\\text{of inside}}}$

this is a composition of functions

$\int_1^x t^3 dt$ composed w/ $x^2$

so use chain rule

(4) $\frac{d}{dx}\int_{e^x}^1 \sin t\,dt = \frac{d}{dx} - \int_1^{e^x} \sin t\,dt$

$= -(\sin e^x)\cdot e^x$

(5) $\frac{d}{dx}\underbrace{x \cdot \int_2^x e^t\,dt}_{\text{product rule}} = \underbrace{\int_2^x e^t\,dt}_{\text{use part II}} + x \cdot e^x$

$= e^x - e^2 + xe^x$

(6) $\int_1^2 3x^2 + 5x + 2\,dx = x^3 + \frac{5x^2}{2} + 2x \Big|_1^2 = (2)^3 + \frac{5(2)^2}{2} + 2(2) - (1 + \frac{5}{2} + 2)$

$= 8 + 10 + 4 - 1 - \frac{5}{2} - 2 = \frac{33}{2}$